Oscillation of partial population model with diffusion and delay

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\textbf{A B S T R A C T}

By using the upper- and lower-solution method of partial functional differential equations and the oscillation theory of functional differential equation, the oscillation of a population equation with diffusion and delay is studied and a sufficient condition for all positive solutions of the equation to oscillate about the positive equilibrium is obtained. Finally, a model arising from ecology is given to illustrate the obtained results.

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1. Introduction

In this paper, the oscillation properties of the solutions are studied in the following nonlinear parabolic system

\[
\frac{\partial u (x, t)}{\partial t} = d(t) \Delta u (x, t) + u (x, t) \left[ a + bu^m (x, t - \tau) - cu^n (x, t - \tau) \right], \quad (x, t) \in \Omega \times R_+,
\]

with the boundary and initial conditions

\[
\frac{\partial u (x, t)}{\partial n} = 0, \quad (x, t) \in \partial \Omega \times R_+,
\]

\[
u (x, \theta) = \varphi (x, \theta), \quad \varphi (x, \theta) \geq 0, \text{ and } \varphi (x, \theta) \text{ is not identically equal to } 0\]

\[
(x, \theta) \in \Omega \times [-\tau, 0).
\]

Here \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( \partial / \partial n \) denotes the outward normal derivative on \( \partial \Omega \), and \( a, b, c \) are positive constants. \( \tau > 0 \) is a bounded constant representing delay, and \( m, n \) are positive integers with \( m < n \). \( \varphi (x, \theta) \) is a smooth function that denotes the initial condition. \( u (x, t) \) denotes the density of the population, \( d(t) \) denotes the diffusion rate of the species, and \( \Delta \) is the Laplace operator. The model (1)–(3) is the extension of the general Logistic model with delay [1]. The oscillation problem for the ordinary differential system with delay has been extensively investigated in [2–5] and lots of results have been obtained. When the increase rate of the species is a linear function of the density of population, that is \( c = 0 \), \( m = 1 \), Luo [6] studies the oscillation of the positive equilibrium in this system. When the increase rate of the species is a nonlinear function of the density of population, Liao and Li [7] investigate the asymptotic stability...
of system (1)–(3) without delay, and Gourley and Britton [8] study the stability of the solution for this system with delay. However, there is very little result about the oscillation theory of partial differential systems with delay (cf. [6,9]).

The objective of this paper is to investigate the oscillation properties of the solutions of a class of nonlinear reaction–diffusion system with delay. A sufficient condition for all positive solutions of the equation to oscillate about the positive equilibrium is obtained by employing the upper- and lower-solution method of partial functional differential equations and the oscillation theory of functional differential equation. To illustrate our results, an example is also given.

This paper is arranged as follows. In Section 2, conditions for the existence and uniqueness of the positive equilibrium of Eq. (1) are established, the condition for the existence of the global solution of Eqs. (1)–(3) is obtained and conditions for nonexistence of ultimately positive solution or negative solution of a nonlinear differential inequality with delay are also given. In Section 3, a sufficient condition for all positive solutions of the equation to oscillate about the positive equilibrium is established. In Section 4, an example is given to illustrate our results.

2. Preliminaries

In this section, we first establish the existence and uniqueness conditions of the positive equilibrium of Eq. (1), and then give the condition for the existence of the global solution of Eqs. (1)–(3) and conditions for nonexistence of ultimately positive solution or negative solution of the nonlinear differential inequality with delay.

Lemma 2.1. Eq. (1) has a unique positive equilibrium \( u^* \), that is

\[ a + bu^m - cu^n = 0, \]

and \( a + bu^m - cu^n < 0 \) when \( u > u^* \); \( a + bu^m - cu^n > 0 \) when \( 0 < u < u^* \).

Proof. Since \( a > 0, \ c > 0 \), Eq. (1) has a unique positive equilibrium by the knowledge of mathematical analysis, and we have

\[ a + bu^m - cu^n < 0, \quad u \in (u^*, +\infty), \]

and

\[ a + bu^m - cu^n > 0, \quad u \in (0, u^*). \]

Definition 2.1. If for any \( T > 0 \) there exists \( (x_0, t_0) \in \Omega \times [T, +\infty) \) such that \( u(x_0, t_0) = u^* \), then we say that the solution \( u(x, t) \) in \( \Omega \times R_+ \) of Eqs. (1)–(3) oscillates about the positive equilibrium \( u^* \), otherwise, we say that \( u(x, t) \) does not oscillate about \( u^* \).

Definition 2.2. Two functions \( \hat{u}(x, t), \hat{u}(x, t) \) in \( Q_T = \Omega \times (−T, T) \) are called a couple of upper and lower solutions of Eqs. (1)–(3), if \( \hat{u}(x, t) \geq \hat{u}(x, t) \) in \( Q_T \) and if

\[ \frac{\partial \hat{u}}{\partial t} - d(t)\Delta \hat{u} \geq \hat{u}(a + bu^m - cu^n), \quad \text{for all } w \in (\hat{u}, \hat{u}), \]

\[ \frac{\partial \hat{u}}{\partial t} - d(t)\Delta \hat{u} \leq \hat{u}(a + bu^m - cu^n), \quad \text{for all } w \in (\hat{u}, \hat{u}), \]

\[ \frac{\partial \hat{u}}{\partial n} \geq 0, \quad \frac{\partial \hat{u}}{\partial n} \leq 0, \quad (x, t) \in \partial \Omega \times (0, T], \]

\[ \hat{u}(x, t) \leq u(x, t), \quad (x, t) \in \Omega \times [−T, 0), \]

where \( (\hat{u}, \hat{u}) \equiv \{ u \in C(Q_T) : \hat{u} \leq u \leq \hat{u} \} \).

Lemma 2.2. If \( u(x, \theta) = \psi(x, \theta) \geq 0, \) and \( \psi(x, \theta) \) is not identically equal to \( 0 \) when \( (x, \theta) \in \Omega \times [−T, 0) \), then Eqs. (1)–(3) have a unique positive global solution in \( Q_T = \Omega \times (−T, \infty) \).

Proof. Since \( a > 0, \ c > 0 \) and \( m < n \), without loss of generality we may suppose that the maximum of the function \( f(u) = a + bu^m - cu^n \) is equal to \( M(M \geq a > 0) \) when \( u > 0 \). Let \( \tilde{u} = M^*e^{M^*(t+\tau)}, \) \( \tilde{u} = 0, \) where \( M^* \geq \max_{(x, \theta) \in \Omega \times [−T, 0)} \psi(x, \theta) \).

It is easy to verify that for any \( T > 0 \) two functions \( \hat{u}, \tilde{u} \) in \( Q_T = \Omega \times (−T, T) \) are a couple of upper and lower solutions of Eqs. (1)–(3). By Theorem 2.2 of [10] Eqs. (1)–(3) have a unique solution \( u(x, t) \) in \( Q_T \) and \( u(x, t) \in [0, \tilde{u} \). Since \( T \) is arbitrary and \( \psi(x, \theta) \) is not identically equal to \( 0 \), Eqs. (1)–(3) have a continuously unique positive global solution \( u(x, t) \) in \( Q_T = \Omega \times (−T, \infty) \).

The following lemma is from [11, P84].

Lemma 2.3. We suppose that
By integrating, it is easy to prove that

Then

3. All positive solutions to oscillate about the equilibrium

In this section, a sufficient condition for all positive solutions of Eqs. (1)-(3) to oscillate about the positive equilibrium is established.

Theorem 3.1. If the following conditions hold

where \( M = \lim_{y \to 0} \frac{y}{f(y)} \).

Then (1) the nonlinear differential inequality with delay \( y'(t) + p(t)f(y(t - \tau)) \leq 0 \) does not have ultimately positive solution;

(2) the nonlinear differential inequality with delay \( y'(t) + p(t)f(y(t - \tau)) \geq 0 \) does not have ultimately negative solution.

(1) \( f \in \mathbb{C}[R, R] \), \( yf(y) > 0 (y \neq 0) \);
(2) \( \lim_{t \to +\infty} \int_{t-\tau}^{t} p(s)ds > \frac{b\tau}{e^\tau} \), where \( M = \lim_{y \to 0} \frac{y}{f(y)} \).

Proof. Let \( u(x, t) \) be a non-oscillatory solution of Eqs. (1)-(3), that is, there exists \( t_0 \geq T \) such that \( u(x, t) > u^* \) or \( u(x, t) < u^* \) when \( (x, t) \in \Omega \times [T, +\infty) \).

Firstly, we consider the case 1: \( u(x, t) > u^* \) when \( (x, t) \in \Omega \times [T, +\infty) \). Without loss of generality we may suppose \( u(x, t - \tau) > u^* \).

Let \( M(x, t) = u(x, t) - u^* > 0, (x, t) \in \Omega \times [T, +\infty) \), then we have

\[
\frac{\partial u(x, t)}{\partial t} = \frac{\partial M(x, t)}{\partial t}, \quad \Delta u(x, t) = \Delta M(x, t).
\]

Substitute \( M(x, t) = u(x, t) - u^* \) into Eq. (1). Noting (4) and \( a + bu^* - cu^* < 0 \) when \( u > u^* \) as well as \( cu^{(n-m)} - b > 0 \), we have

\[
\frac{\partial M(x, t)}{\partial t} = d(t) \Delta M(x, t) + M(x, t) \left\{ a + b \left[ M(x, t - \tau) + u^* \right]^m - c \left[ M(x, t - \tau) + u^* \right]^n \right\} + u^* \left\{ a + b \left[ M(x, t - \tau) + u^* \right]^m - c \left[ M(x, t - \tau) + u^* \right]^n \right\} < d(t) \Delta M(x, t) + u^* (bM^m(x, t - \tau) + bc^{m-1}_m M^{m-1}(x, t - \tau) u^* + \cdots \right.
\]

\[
+ bc^{m-2}_m M^2(x, t - \tau) u^{(m-2)} + bc^{m-1}_m M(x, t - \tau) u^{(m-1)} - c c^{m-1}_n M^{m-1}(x, t - \tau) u^{(n-m-1)} - \cdots - c c^{m-2}_n M^2(x, t - \tau) u^{(n-m)} - \cdots \right.
\]

\[
- c c^{m-2}_n M^2(x, t - \tau) u^{(n-m-2)} - c c^{m-1}_n M(x, t - \tau) u^{(n-m-1)} u^* + \cdots \right.
\]

Since

\[
\begin{align*}
&b c^{m-1}_n M^{m-1}(x, t - \tau) u^{(n-m-1)} - c c^{m-1}_n M^{m-1}(x, t - \tau) u^{(n-m-1)} - \cdots - c c^{m-2}_n M^2(x, t - \tau) u^{(n-m-2)} \\
&= \sum_{k=0}^{m-2} [bc^{m-k}_m c^{m-k}(x, t - \tau) u^{(n-m-k)}] \\
&= c^{m-2}_m M^{m-2}(x, t - \tau) u^{(n-m-2)} - c c^{m-1}_n M^{m-1}(x, t - \tau) u^{(n-m-1)} - \cdots - c c^{m-2}_n M^2(x, t - \tau) u^{(n-m)} - \cdots \right.
\end{align*}
\]

It is easy to prove that \( c^{m}_m < c^{m-1}_n \). It follows from condition (5) that \( b - c u^{(n-m)} < 0 \). This ensures that

\[
\frac{\partial M(x, t)}{\partial t} < d(t) \Delta M(x, t) + u^* \left[ b u^{(m-1)} - c u^{(n-1)} \right] M(x, t - \tau).
\]

By integrating (6) about \( x \) in \( \Omega \), we have

\[
\frac{\partial}{\partial t} \int_{\Omega} M(x, t)dx < d(x) \int_{\Omega} \Delta M(x, t)dx + u^* \left[ b u^{(m-1)} - c u^{(n-1)} \right] \int_{\Omega} M(x, t - \tau)dx.
\]
It follows from the "Green Formula" and boundary condition (2) that
\[
\int_{\Omega} \Delta M(x, t) \, dx = \int_{\partial \Omega} \frac{\partial M}{\partial n} \, ds = 0.
\] (8)

Hence, let \( v(t) = \frac{1}{|\Omega|} \int_\Omega M(x, t) \, dx \), \( M(x, t) \) and \( v(t) \), \( t \geq T \), then \( v(t) > 0 \). From (7) and (8), we have
\[
v'(t) + u^* (c m u^{(n-1)} - b m u^{(n-1)}) v(t - \tau) < 0.
\] (9)

That is, \( v(t) \) is the ultimately positive solution of inequality (9).

However, condition (5) gives \( (c m u^{(n-m)} - b u^{(n-m)}) \tau > \frac{1}{e} \), and it follows from Lemma 2.3 that differential inequality with delay \( (9) \) does not have ultimately positive solution. This is a contradiction. Therefore, case 1 does not hold.

Secondly, we consider the case \( 2 : 0 < u(x, t) < u^* \) when \( (x, t) \in \Omega \times [T, +\infty) \). Without loss of generality we may also suppose \( 0 < u(x, t - \tau) < u^* \). Let \( u(x, t) = u^* e^{w(x,t)} \), then \( u(x, t) < 0 \) and \( u(x, t - \tau) < 0 \). By substituting \( u(x, t) = u^* e^{w(x,t)} \) into Eq. (1) and using (4) and the condition \( c u^{(n-m)} - b > 0 \), \( (m \geq 2) \), we have
\[
\frac{\partial w(x, t)}{\partial t} = d(t) [\Delta w(x, t) + |\nabla w(x, t)|^2] + [a + b u^{(m-1)} e^{u(x,t-\tau)} - c u^{(n-1)} e^{u(x,t-\tau)}]
\]
\[
= d(t) [\Delta w(x, t) + |\nabla w(x, t)|^2] + [e^{w(x,t-\tau)} - 1] [(b u^{(m-1)} - c u^{(n-1)}) e^{(m-1)w(x,t-\tau)} + e^{(m-2)w(x,t-\tau)} + \cdots + e^{(m-n+1)w(x,t-\tau)} + 1]].
\]

Using the fact that \( b u^{(m-1)} - c u^{(n-1)} < 0 \) due to \( c u^{(n-m)} - b > 0 \), one gets
\[
\frac{\partial w(x, t)}{\partial t} > d(t) \Delta w(x, t) + [e^{w(x,t-\tau)} - 1] [b u^{(m-1)} - c u^{(n-1)} e^{(m-1)w(x,t-\tau)} + 1]].
\]

Since \( a + b u^{(m-1)} - c u^{(n-1)} = 0 \), hence
\[
\frac{\partial w(x, t)}{\partial t} > d(t) \Delta w(x, t) - [a + c u^{(n-1)} e^{(n-1)w(x,t-\tau)}] [e^{w(x,t-\tau)} - 1].
\] (10)

By integrating (10) about \( x \) in \( \Omega \), we have
\[
\frac{\partial}{\partial t} \int_{\Omega} w(x, t) \, dx + \int_{\Omega} (a + c u^{(n-1)} e^{(n-1)w(x,t-\tau)}) (e^{w(x,t-\tau)} - 1) \, dx > 0.
\]

And then, let \( v(t) = \frac{1}{|\Omega|} \int_{\Omega} w(x, t) \, dx < 0 \), \( t \geq T \), then \( v(t) \) is the ultimately negative solution of the differential inequality with delay
\[
v'(t) + f(v(t - \tau)) > 0,
\] (11)
where \( f(v(t)) = \frac{1}{|\Omega|} \int_{\Omega} (a + c u^{(n-1)} e^{(n-1)w(x,t)}) (e^{w(x,t)} - 1) \, dx \).

However, since
\[
\lim_{y \to 0} \frac{y}{f(y)} = \frac{1}{a + c u^{(n-1)}}
\]
and
\[
a + c u^{(n-m)} = (2 c u^{(n-m)} - b) u^{(m-n)},
\]
it follows from condition (5) and Lemma 2.3 that the differential inequality with delay (11) does not have ultimately negative solution. This is a contradiction.

Therefore, case 2 does not hold too. And then, all positive solutions of Eqs. (1)-(3) oscillate about the positive equilibrium. □

When \( m = 1, \ n = 2 \), we have

**Corollary 3.1.** If the following condition holds
\[
(2 c u^* - b) u^* \tau > \frac{1}{e},
\]
then all positive solutions of Eqs. (1)-(3) oscillate about the positive equilibrium.

When \( m = 2, \ n = 3 \), we have
Corollary 3.2. If the following condition holds
\[(cu^* - b)u^{*2} \tau > \frac{1}{e},\]
then all positive solutions of Eqs. (1)–(3) oscillate about the positive equilibrium.

4. Applications and examples

In this section, an example is given to illustrate our results.

Example: We consider the following partial population model with delay and diffusion
\[
\frac{\partial u(x, t)}{\partial t} = \Delta u(x, t) + u(x, t) \left[ 1 + u^3(x, t - \frac{1}{e}) - 3u^4(x, t - \frac{1}{e}) \right], \quad (x, t) \in \Omega \times R_+,
\]
with the boundary and initial conditions
\[
\frac{\partial u(x, t)}{\partial n} = 0, \quad (x, t) \in \partial \Omega \times R_+,
\]
\[
u(x, \theta) = \psi(x, \theta), \quad \psi(x, \theta) \geq 0, \quad \text{and} \quad \psi(x, \theta) \text{ is not identically equal to 0}
\]
\[(x, \theta) \in \Omega \times [-\tau, 0).\]

It is obvious that Eq. (12) has a unique positive equilibrium \(u^* = 1\). For any initial \(\psi(x, \theta) \geq 0\) such that \(\psi(x, \theta) \) is not identically equal to 0, it follows from Lemma 2.2 that Eqs. (12)–(14) have a unique continuously positive global solution \(u(x, t)\). It is obvious that Eqs. (12)–(14) satisfy the conditions of Theorem 3.1.

In fact, \(a = 1, b = 1, c = 3, d(t) = 1, \tau = \frac{1}{e}, m = 3, n = 4\).

Thus,
\[
(2cu^{(n-m)} - bm)u^{*m} \tau = 3 \times \frac{1}{e} > \frac{1}{e}.
\]

It follows from Theorem 3.1 that all positive solutions of this equation oscillate about the positive equilibrium \(u^* = 1\).

Remark: It is very obvious that the results of [6] cannot validate the oscillation of Eqs. (12)–(14).

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References