On starshaped fuzzy sets

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Abstract
In this paper we firstly define some new types of fuzzy starshapedness, discuss the relationships among the different types of fuzzy starshapedness, and then present some basic properties of them. Finally we develop several important results on the shadows of starshaped fuzzy sets.

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1. Introduction
Convexity plays a most useful role in the theory and applications of fuzzy sets. In the basic and classical paper [1], Zadeh paid special attention to the investigation of the convex fuzzy sets. Following the seminal work of Zadeh on the definition of a convex fuzzy set, Ammar and Metz defined another type of convex fuzzy sets in [2]. From then on, Zadeh’s convex fuzzy sets were called quasi-convex fuzzy sets in order to avoid misunderstanding. A lot of scholars have discussed various aspects of the theory and applications of quasi-convex fuzzy sets and convex fuzzy sets [3–11].
However, nature is not convex and, apart from possible applications, it is of independent interest to see how far the supposition of convexity can be weakened without losing too much structure. Starshaped sets are a fairly natural extension which is also an important issue in classical convex analysis [12–15]. Many remarkable theorems such as Helly’s Theorem [16] and Krasnosel’skii’s Theorem [17] relate to starshaped sets. In [3], Brown introduced the concept of starshaped fuzzy sets, and in [18] Diamond defined another type of starshaped fuzzy sets and established some of the basic properties of this family of fuzzy sets. To avoid misunderstanding, Brown’s starshaped fuzzy sets will be called quasi-starshaped fuzzy sets in the sequel. Recently, the research of fuzzy starshaped (f.s.) set has been again attracting the deserving attention [19–21]. But with regards to Diamond’s definition, there exists a small mistake which has been corrected in [21]. Motivated both by Diamond’s research and by the importance of the concept of fuzzy convexity, we propose in this paper new and more general definitions in the area of fuzzy starshapedness.
Shadow of fuzzy set is another important concept in the classical paper [1]. In [1,22] Zadeh, and in [7] Liu obtained some interesting results on the shadows of convex fuzzy sets. Recently, some authors made further investigation about this subject [23,24]. Inspired by Liu’s work [7], we will present fundamental discussion about shadows of starshaped fuzzy sets. Thus our paper is a natural continuation of the articles [7,18] concerning the properties of starshaped fuzzy set.

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In sum, our purpose is fourfold. Firstly, we introduce fuzzy general starshaped (f.g.s.) sets and fuzzy pseudo-starshaped (f.p.s.) sets which are derived from the concepts of quasi-convex fuzzy sets [1] and convex fuzzy sets [2]. Secondly, we clarify the exact relationships among the concepts of f.s. sets [18], fuzzy quasi-starshaped (f.q.s.) sets [3], f.p.s. sets and f.g.s. sets. Thirdly, we give a detailed study on the basic properties of these different types of starshapedness (see Propositions 1–18). Finally, we investigate the shadows of fuzzy sets and derive several important results on the shadows of starshaped fuzzy sets.

For simplicity, we consider only the starshaped fuzzy sets defined on the Euclidean space in this paper. But it is not difficult to generalize most of the results obtained in the paper to the case that starshaped fuzzy sets are defined in linear space over real field or complex field.

The paper is divided into seven sections. In Section 2, after recalling starshapedness in $\mathbb{R}^n$ and the basic definition of starshaped fuzzy set in [18], we introduce some new different types of starshapedness. In Section 3, we discuss the relationships among these different types of starshapedness. In Section 4, we study some important properties of these different types of starshapedness. In Section 5, we derive several new results on the shadows of starshaped fuzzy sets. In Section 6, we give some applications of fuzzy starshapedness. Finally in Section 7, we conclude this paper with some short remarks.

2. Notation and definitions

Let $x, y \in \mathbb{R}^n$, the line segment $\overline{xy}$ joining $x$ and $y$ is the set of all points of the form $zx + \beta y$ where $z \geq 0, \beta \geq 0$, $z + \beta = 1$. A set $S \subseteq \mathbb{R}^n$ is said to be starshaped relative to a point $x \in \mathbb{R}^n$ if for each point $y \in S$, it is true that $\overline{xy} \subseteq S$. A set $S$ is simply said to be starshaped, which means that there is some point $x$ in $\mathbb{R}^n$ such that $S$ is starshaped relative to it. The kernel $\ker S$ of $S$ is the set of all points $x \in S$ such that $\overline{xy} \subseteq S$ for each $y \in S$ and $\text{conv } S$ is the convex hull of $S$. The convex hull $A$ of a finite set of $r + 1$ points $x_1, x_2, \ldots, x_{r+1} \in \mathbb{R}^n$ is called an $r$-dimensional simplex if the linear subspace of minimal dimension containing $A$ has dimension $r$. The points $x_i, i = 1, \ldots, r + 1$ are called vertices. Most results on ordinary convex sets which are used in this paper can be found in [25,26].

Let $\mathcal{F}(\mathbb{R}^n)$ denote the class of normal fuzzy sets on $\mathbb{R}^n$, that is, $\mu \in \mathcal{F}(\mathbb{R}^n)$ is a mapping $\mu : \mathbb{R}^n \rightarrow [0, 1]$, such that $\{x \in \mathbb{R}^n : \mu(x) = 1\}$ is nonempty. An $x$-level set of $\mu$ is $[\mu]^x = \{x : \mu(x) \geq x\}$, for $0 \leq x \leq 1$. The support set of $\mu$ is $\text{supp}(\mu) = \{x : \mu(x) > 0\}$, where $\{x : \mu(x) > 0\}$ is the closure of set $\{x : \mu(x) > 0\}$. In [2] a fuzzy set $\mu$ is quasi-convex if

$$\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in \mathbb{R}^n$, $0 \leq \lambda \leq 1$, and a fuzzy set $\mu$ is called convex if

$$\mu(\lambda x + (1 - \lambda)y) \geq \lambda \mu(x) + (1 - \lambda)\mu(y)$$

for all $x, y \in \text{supp}(\mu)$, $0 \leq \lambda \leq 1$.

Diamond’s definition of f.s. set can be expressed as follows:

**Definition 2.1 (Diamond [18]).** A fuzzy set $\mu \in \mathcal{F}(\mathbb{R}^n)$ is said to be f.s. with respect to $y \in \mathbb{R}^n$ if, for all $x \in \mathbb{R}^n$,

$$\mu(\lambda(x - y)) \geq \mu(x - y), \quad 0 \leq \lambda \leq 1.$$ 

In [18] Diamond asserted that $\mu \in \mathcal{F}(\mathbb{R}^n)$ is f.s. with respect to $y$ iff its level sets are starshaped with respect to $y$. The following counterexample will show that the above assertion is incorrect.

**Example 1.** Define $\mu \in \mathcal{F}(\mathbb{R})$ as follows:

$$\mu(x) = \begin{cases} 
  x + 1, & x \in [-1, 0), \\
  1 - x, & x \in [0, 1], \\
  0 & \text{otherwise}.
\end{cases}$$

Let $y = 2$. For all $x \in \mathbb{R}$, by a simple calculation we have $\mu(\lambda(x - 2)) \geq \mu(x - 2), 0 \leq \lambda \leq 1$. However, all the level sets $[\mu]^x$ with $x > 0$ are not starshaped relative to $y = 2$ since $2 \notin [\mu]^x = [-1 + x, 1 - x]$. 

In order to recover the assertion, f.s. set has been redefined in [21] as follows:

**Definition 2.2 (Wu and Zhao [21])**. A fuzzy set \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is said to be f.s. relative to \( y \in \mathbb{R}^n \) if, for all \( x \in \mathbb{R}^n \),

\[
\mu(\lambda(x - y) + y) \geq \mu(x), \quad 0 \leq \lambda \leq 1.
\]

**Proposition 1.** \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is f.s. with respect to \( y \) iff its level sets are starshaped with respect to \( y \).

**Proof.** Suppose the \( z \)-level set \([\mu]^z\) is starshaped relative to \( y \) for all \( z \in [0, 1] \). For \( x \in \mathbb{R}^n \), let \( z = \mu(x) \). Then \( \overline{xy} \subseteq [\mu]^z \), that is, \( \mu(\lambda(x - y) + y) \geq \mu(x) \), \( 0 \leq \lambda \leq 1 \). Conversely, if for all \( x \in \mathbb{R}^n \), \( \mu(\lambda(x - y) + y) \geq \mu(x) \), \( 0 \leq \lambda \leq 1 \). Since \([\mu]^1 \neq \emptyset \), we suppose \( x \in [\mu]^z \), that is, \( \mu(x) \geq z \). Hence \( \mu(\lambda(x - y) + y) \geq \mu(x) \geq z \) for \( \lambda \in [0, 1] \), that is, \( \overline{xy} \subseteq [\mu]^z \).

Thus \([\mu]^z\) is starshaped relative to \( y \). We complete the proof here. \( \square \)

Next, we extend the starshapedness as follows:

**Definition 2.3.** A fuzzy set \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is said to be f.g.s. if all level sets are starshaped sets in \( \mathbb{R}^n \).

Equivalently, a fuzzy set \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is said to be f.g.s. if its each level set \([\mu]^z\) is starshaped relative to a point \( x_z \in \mathbb{R}^n \) for \( 0 \leq z \leq 1 \).

Two more other types of starshapedness are derived from the concepts of quasi-convex fuzzy sets and convex fuzzy sets.

**Definition 2.4 (Brown [3]).** A fuzzy set \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is said to be f.q.s. relative to \( y \in \mathbb{R}^n \) if, for all \( x \in \mathbb{R}^n \),

\[
\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}, \quad 0 \leq \lambda \leq 1.
\]

**Definition 2.5.** A fuzzy set \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is said to be f.p.s. relative to \( y \in \text{supp}(\mu) \) if, for all \( x \in \text{supp}(\mu) \),

\[
\mu(\lambda x + (1 - \lambda)y) \geq \lambda \mu(x) + (1 - \lambda)\mu(y), \quad 0 \leq \lambda \leq 1.
\]

**Definition 2.6.** Let \( \text{ker}(\mu) \) (respectively, \( \text{p-ker}(\mu) \), \( \text{q-ker}(\mu) \)) be the totality of \( y \in \mathbb{R}^n \) such that \( \mu \) is f.s. (respectively, f.p.s., f.q.s.) relative to \( y \).

Since in this paper the concept of support set of a fuzzy set is different from the one in [2], we modify the definition of the fuzzy hypograph in [2] as follows.

**Definition 2.7.** The fuzzy hypograph of \( \mu \) denoted by \( f\text{.hyp}(\mu) \), is defined as

\[ f\text{.hyp}(\mu) = \{(x, t) : x \in \text{supp}(\mu), t \in [0, \mu(x)]\}. \]

**Definition 2.8 (Liu [7]).** Let \( H \) be the ordinary hyperplane of the Euclidean space \( \mathbb{R}^n \). The orthogonal projection \( P : \mathbb{R}^n \to H \) induces a correspondence \( S_H \) from \( \mathcal{F}(\mathbb{R}^n) \) into \( \mathcal{F}(H) \). Then for each fuzzy set \( \mu \) on \( \mathbb{R}^n \), the image \( S_H(\mu) \) is called the shadow of \( \mu \) on \( H \).

**Remark 1 (Liu [7]).** The shadow \( S_H(\mu) \) can be expressed as \( S_H(\mu)(y) = \sup\{\mu(x) : x \in \mathbb{R}^n \text{ and } P(x) = y\} \).

**Definition 2.9 (Lowen [8]).** If \( \mu \) is a fuzzy set then define its convex hull as

\[ \text{conv}(\mu) = \inf\{v : v \supseteq \mu, v \text{ is quasi-convex}\} = \text{smallest quasi-convex set containing } \mu. \]

3. **Relationships on starshapedness of fuzzy sets**

In this section, we discuss the relationships among these different types of starshapedness.
Theorem 3.1. For a fuzzy set $\mu \in \mathcal{F}(\mathbb{R}^n)$, the following statements hold:

(i) If $\mu$ is f.s. relative to $y$, then it is f.g.s.;
(ii) If $\mu$ is f.g.s. and the intersection of all $\ker[\mu]^z$ is nonempty, then it is f.s. relative to some point in $\mathbb{R}^n$;
(iii) If $\mu$ is f.s. relative to $y$, then it is f.q.s. relative to the same point;
(iv) If $\mu$ is f.p.s. relative to $y$, then it is f.q.s. relative to the same point;
(v) If $\mu$ is f.q.s. relative to $y$ and $\mu(y) = \sup_{x \in \mathbb{R}^n} \mu(x) = 1$, then it is f.s. relative to the same point;
(vi) If $\mu$ is f.p.s. relative to $y$ and $\mu(y) = \sup_{x \in \mathbb{R}^n} \mu(x) = 1$, then it is f.s. relative to the same point.

Proof. (i) It follows quickly from Definition 2.3 and Proposition 1.
(ii) Since $\bigcap_{x \in [0,1]} \ker[\mu]^z \neq \emptyset$, there is a point $y$ in the intersection. Thus by Proposition 1 we get that $\mu$ is f.s. relative to $y$.
(iii) For all $x \in \mathbb{R}^n$, $\mu(\lambda(x - y) + y) \geq \mu(x), 0 \leq \lambda \leq 1$, putting $\lambda = 0$ we get $\mu(y) \geq \mu(x)$ for all $x \in \mathbb{R}^n$. Thus $\mu(\lambda(x - y) + y) \geq \mu(x) = \min\{\mu(x), \mu(y)\}$ for $\lambda \in [0, 1]$.
(iv) It follows from Definition 2.4 since for all $x \in \mathbb{R}^n$, $\mu(\lambda x + (1 - \lambda)y) \geq \lambda \mu(x) + (1 - \lambda)\mu(y) \geq \min\{\mu(x), \mu(y)\}$ for $\lambda \in [0, 1]$.
(v) If $\mu$ is f.q.s. relative to $y$ and $\mu(y) = \sup_{x \in \mathbb{R}^n} \mu(x)$, then for all $x \in \mathbb{R}^n$, $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\} = \mu(x)$, $0 \leq \lambda \leq 1$.
(vi) It follows from (iv) and (v). We complete the whole proof here. □

Remark 2. The inverse statements of (i), (iii), (iv) and (vi) do not hold in general as shown in the following examples.

Example 2. Let

$$A = \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 0, -y \leq x \leq (2 - y)/2 \text{ or } 0 \leq y \leq 2, y \leq x \leq (2 + y)/2\},$$
$$B = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2, |y| \leq x \text{ or } 2 \leq x \leq 4, 2x - 8 \leq y \leq 4 - x\},$$
$$C = B \setminus A = \{(x, y) \in \mathbb{R}^2 : (x, y) \in B \text{ but } (x, y) \notin A\},$$
$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2/3, |y| \leq x \text{ or } 2/3 \leq x \leq 1, |y| \leq 2(1 - x)\}$$

and

$$E = \{(x, y) \in \mathbb{R}^2 : 2 \leq x \leq 8/3, -x \leq y \leq 4 - x \text{ or } 8/3 \leq x \leq 4, 2x - 8 \leq y \leq 4 - x\}.$$

Define the fuzzy set $\mu \in \mathcal{F}(\mathbb{R}^2)$ as

$$\mu((x, y)) = \begin{cases} 1, & (x, y) \in A, \\ 0.5, & (x, y) \in C, \\ 0, & \text{otherwise}. \end{cases}$$

Now we can calculate all the kernels of $[\mu]^z$ as follows:

$$\ker[\mu]^z = \begin{cases} D = \ker A, & 0.5 < x \leq 1, \\ E = \ker B, & 0 < x \leq 0.5, \\ \mathbb{R}^2, & x = 0. \end{cases}$$

Then $\mu$ is f.g.s. But it is not f.s. since $D \cap E = \emptyset$.

Example 3. The fuzzy set $\mu \in \mathcal{F}(\mathbb{R})$ with

$$\mu(x) = \begin{cases} x + 1.5, & x \in [-1.5, -0.5], \\ 0.5 - x, & x \in (-0.5, 0], \\ x + 0.5, & x \in [0, 0.5], \\ 1.5 - x, & x \in (0.5, 1.5], \\ 0, & \text{otherwise} \end{cases}$$

is f.p.s. and f.q.s. relative to $y = 0$. But it is not f.s. relative to $y = 0$. 

Example 4. The fuzzy set $\mu \in \mathcal{F}(\mathbb{R})$ with
\[
\mu(x) = \begin{cases} 
2 + x, & x \in [-2, -1], \\
x^2, & x \in (-1, 1], \\
2 - x, & x \in [1, 2], \\
0 & \text{otherwise}
\end{cases}
\]
is f.q-s. relative to $y = 0$, but it is not f.p-s. relative to the same point.

Example 5. The fuzzy set $\mu \in \mathcal{F}(\mathbb{R})$ with
\[
\mu(x) = \begin{cases} 
e^x, & x \in (-\infty, 0], \\
-x, & x \in (0, +\infty)
\end{cases}
\]
is f.s. relative to $y = 0$, but it is not f.p-s. relative to the same point.

4. Basic properties of starshapedness of fuzzy sets

In this section, some basic properties of starshapedness of fuzzy sets are discussed.

Proposition 2. Let $\mu \in \mathcal{F}(\mathbb{R}^n)$ be f.s. relative to $y \in \mathbb{R}^n$. Then $\mu(y) = \sup_{x \in \mathbb{R}^n} \mu(x) = 1$.

Proof. By Definition 2.2, we have that for all $x \in \mathbb{R}^n$, $\mu(\lambda(x - y) + y) \geq \mu(x)$, $0 \leq \lambda \leq 1$. Putting $\lambda = 0$ we get $\mu(y) \geq \mu(x)$ for all $x \in \mathbb{R}^n$. Then $\mu(y) = \sup_{x \in \mathbb{R}^n} \mu(x) = 1$. □

For a starshaped fuzzy set $\mu \in \mathcal{F}(\mathbb{R}^n)$, if for any $\alpha, \beta \in [0, 1]$, $\alpha \leq \beta$, we have that $\text{ker}[\mu]^z \supseteq \ker[\mu]^\beta$. Define a fuzzy set $f\text{ker}(\mu)$ by $[f\text{ker}(\mu)]^z = \ker[\mu]^z$.

Proposition 3. For a starshaped fuzzy set $\mu \in \mathcal{F}(\mathbb{R}^n)$, $f\text{ker}(\mu)$ is a quasi-convex fuzzy set.

Proof. Since its level sets, $\ker[\mu]^z$, are convex [26], we have that $f\text{ker}(\mu)$ is quasi-convex. □

Proposition 4. Let $\mu \in \mathcal{F}(\mathbb{R}^n)$. Then $\ker(\mu)$ is a convex set in $\mathbb{R}^n$ and $\ker(\mu) \subseteq \ker[\mu]^z$ for all $z \in [0, 1]$.

Proof. If $\ker(\mu) = \emptyset$, it is obvious that $\ker(\mu)$ is convex. Suppose $\ker(\mu) \neq \emptyset$. Let $y_1, y_2 \in \ker(\mu)$. From Proposition 1, we have its level sets are starshaped relative to $y_1$ and $y_2$. That is, $y_1, y_2 \in \ker[\mu]^z$ for $z \in [0, 1]$. Since $\ker[\mu]^z$ are convex, we have $\overline{\gamma_1 \gamma_2} \subseteq \ker[\mu]^z$ for $z \in [0, 1]$. Thus $\overline{\gamma_1 \gamma_2} \subseteq \ker(\mu)$. □

Remark 3. This statement does not hold for f.q-s. sets or f.p-s. sets. In Example 3, we can get that $q\ker(\mu) = (-\infty, -1] \cup [0] \cup [1, +\infty)$ which is not convex.

Example 6. Define the fuzzy set $\mu \in \mathcal{F}(\mathbb{R})$ as
\[
\mu(x) = \begin{cases} 
x + 1.5, & x \in [-1.5, -0.5], \\
(4 - 2x)/5, & x \in (-0.5, 0], \\
(4 + 2x)/5, & x \in [0, 0.5], \\
1.5 - x, & x \in (0.5, 1.5], \\
0 & \text{otherwise.}
\end{cases}
\]

Then $\mu$ is an f.p-s. set but its $p\ker(\mu) = [-1.5, -3.5/3] \cup [0] \cup [3.5/3, 1.5]$ is not convex.

Proposition 5. $\mu \in \mathcal{F}(\mathbb{R}^n)$ is f.s. with respect to $y$ iff for all $x \in \mathbb{R}^n$,
\[
\mu(\lambda x + y) \geq \mu(x + y), \quad 0 \leq \lambda \leq 1.
\]
Proof. Suppose \( \mu \) is f.s. relative to \( y \), that is, for all \( x \in \mathbb{R}^n \),
\[
\mu(\lambda(x - y) + y) \geq \mu(x), \quad 0 \leq \lambda \leq 1.
\]
Replacing \( x \) by \( x + y \) in the above inequality, we get the desired result. Similarly, we can get the converse. \( \square \)

**Proposition 6.** \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is f.q.s. with respect to \( y \) iff its \( x \)-level sets are starshaped with respect to \( y \) for \( x \in [0, \mu(y)] \).

**Proof.** Suppose \( \mu \) is f.q.s. with respect to \( y \), that is, for all \( x \in \mathbb{R}^n \),
\[
\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}, \quad 0 \leq \lambda \leq 1.
\]
For any \( x \in [0, \mu(y)] \), let \( x \in [\mu]^2 \). Then we have that \( x, y \in [\mu]^2 \). From the above inequality we get that \( \mu(\lambda x + (1 - \lambda)y) \geq x \). That is, \( \nabla \subseteq [\mu]^2 \). Conversely, for \( x \in \mathbb{R}^n \), if \( \mu(x) > \mu(y) \), then let \( x = \mu(y) \). Accordingly we have \( \nabla \subseteq [\mu]^2 \), that is,
\[
\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}, \quad 0 \leq \lambda \leq 1.
\]
If \( \mu(x) \leq \mu(y) \), then let \( x = \mu(x) \). Similarly we have
\[
\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}, \quad 0 \leq \lambda \leq 1.
\]
We complete the proof here. \( \square \)

**Proposition 7.** \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is f.p.s. with respect to \( y \) iff its \( f.\text{hyp}(\mu) \) is starshaped relative to \( y, \mu(y) \).

**Proof.** Let \( \mu \) be f.p.s. relative to \( y \) and \( (x, t) \in \text{f.hyp}(\mu) \). Since \( \mu \) is f.p.s. relative to \( y \), for each \( x \in [0, 1] \) we have
\[
\mu(\lambda x + (1 - \lambda)y) \geq \lambda \mu(x) + (1 - \lambda)\mu(y) \geq \lambda t + (1 - \lambda)\mu(y),
\]
and hence
\[
\lambda(x, t) + (1 - \lambda)(y, \mu(y)) \in \text{f.hyp}(\mu).
\]
Conversely, assume that \( x \in \text{supp}(\mu) \) and \( (x, \mu(x)) \in \text{f.hyp}(\mu) \). By the starshapedness of \( \text{f.hyp}(\mu) \) we have
\[
(\lambda x + (1 - \lambda)y, \lambda \mu(x) + (1 - \lambda)\mu(y)) \in \text{f.hyp}(\mu)
\]
for each \( x \in [0, 1] \). Then
\[
\mu(\lambda x + (1 - \lambda)y) \geq \lambda \mu(x) + (1 - \lambda)\mu(y)
\]
for each \( x \in [0, 1] \). We complete the proof here. \( \square \)

Let \( x_0 \) be some point in \( \mathbb{R}^n \) and \( \mu \) some fuzzy set then the translation of \( \mu \) by \( x_0 \) is the fuzzy set \( x_0 + \mu \) defined as \( (x_0 + \mu)(x) = \mu(x - x_0) \) [6].

**Proposition 8.** Suppose that \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is f.s. (respectively, f.p.s., f.q.s.) relative to \( y \) and there is a point \( x_0 \in \mathbb{R}^n \). Then \( x_0 + \mu \) is f.s. (respectively, f.p.s., f.q.s.) relative to \( x_0 + y \).

**Proof.** We only give the proof for the case of fuzzy starshapedness. Similarly, the others can be proved.
For any \( x \in \mathbb{R}^n \), since \( \mu \) is f.s. relative to \( y \) we have that
\[
(x_0 + \mu)(\lambda(x - y - x_0) + y + x_0) = \mu(\lambda(x - x_0 - y) + y) \geq \mu(x - x_0) = (x_0 + \mu)(x).
\]
Hence, \( (x_0 + \mu) \) is f.s. relative to \( x_0 + y \). \( \square \)
Remark 4. Since the property of being starshaped is translation invariant in $\mathbb{R}^n$, this proposition also holds for fuzzy general starshapedness.

Let $T$ be a linear invertible transformation on $\mathbb{R}^n$ and $\mu$ a fuzzy set. Then by the Extension Principle we have that $(T(\mu))(x) = \mu(T^{-1}(x))$.

Proposition 9. Suppose that $\mu \in \mathcal{F}(\mathbb{R}^n)$ is f.s. (respectively, f.p.-s., f.q.-s.) relative to $y$ and $T$ a linear invertible transformation on $\mathbb{R}^n$. Then $T(\mu)$ is f.s. (respectively, f.p.-s., f.q.-s.) relative to $T(y)$.

Proof. We only give the proof for the case of fuzzy quasi-starshapedness. Similarly, the others can be proved.

For any $x \in \mathbb{R}^n$, since $\mu$ is f.q.-s. relative to $y$ we have that

$$(T(\mu))(\lambda x + (1-\lambda)T(y)) = \mu(\lambda T^{-1}(x) + (1-\lambda)y) 
\geq \min\{\mu(T^{-1}(x)), \mu(y)\} 
= \min\{(T\mu)(x), (T\mu)(T(y))\}.$$ 

Hence, $T(\mu)$ is f.q.-s. relative to $T(y)$. $\square$

Remark 5. Since rotation and mirror image transformation are linear invertible transformations, this proposition holds for them. Moreover, the linear invertible transformation invariant of the starshapedness of the ordinary starshaped set in $\mathbb{R}^n$, this proposition also holds for fuzzy general starshapedness.

A path in a set $S$ in $\mathbb{R}^n$ is a continuous mapping $f$ from the interval $[0, 1]$ to $S$. A set $S$ is said to be path connected if, for all $x, y \in S$, there exists a path $f$ such that $f(0) = x$ and $f(1) = y$ [27]. Equivalently, a set $S$ is said to be path connected if any two points in $S$ can be connected by an unbroken curve lying entirely within $S$. A fuzzy set $\mu$ is said to be a path connected fuzzy set if its level sets are path connected [3]. Since a starshaped crisp set is path connected, one can easily prove the following proposition.

Proposition 10. $\mu \in \mathcal{F}(\mathbb{R}^n)$ is f.s. relative to $y$, or is f.g.s., then $\mu$ is a path connected fuzzy set.

Proposition 11. If $\mu \in \mathcal{F}(\mathbb{R}^n)$ is a fuzzy quasi-convex set, then it is f.g.s.; furthermore if $\mu \in \mathcal{F}(\mathbb{R})$ is f.g.s., then $\mu$ is f.s., and is a fuzzy quasi-convex set.

Proof. Since the first statement is trivial, we only prove the latter. Since $\mu \in \mathcal{F}(\mathbb{R})$ is f.g.s., we have that all the level sets are starshaped. Thus they are path connected sets. In other words, they are intervals and there is at least one point $y$ in $[\mu]^1$. Since intervals are convex sets in $\mathbb{R}$ we have that $\mu$ is f.s. relative to $y$, and is a fuzzy quasi-convex set. $\square$

Proposition 12. If for a fuzzy set $\mu \in \mathcal{F}(\mathbb{R}^n)$, the point $y \in \mathbb{R}^n$ satisfies that $\mu(y) = \inf_{x \in \mathbb{R}^n} \mu(x)$. Then $\mu$ is f.q.-s. relative to $y$, that is, $y \in q$-ker($\mu$).

Proof. According to Definition 2.4, this statement is true since $\mu(y) = \inf_{x \in \mathbb{R}^n} \mu(x)$. $\square$

Proposition 13. $\mu \in \mathcal{F}(\mathbb{R}^n)$ is f.q.s. relative to $y$ and $\mu(y) \neq 0$. Then supp($\mu$) is starshaped relative to $y$.

Proof. First, we prove that the set $A = \{x \in \mathbb{R}^n : \mu(x) > 0\}$ is starshaped relative to $y$. Let $x \in A$. If $\mu(x) > \mu(y)$, then let $z = \mu(y)$. By Proposition 6 we have $\overline{xy} \subseteq [\mu]^2 \subseteq A$. If $\mu(x) \leq \mu(y)$, then let $z = \mu(x)$. Similarly we have $\overline{xyz} \subseteq [\mu]^2 \subseteq A$.

Now, we prove that supp($\mu$) = $\overline{A}$ is starshaped relative to $y$. Let $x_0 \in$ supp($\mu$). Choose $\lambda x_0 + (1-\lambda)y$ with $\lambda \in [0, 1]$. Since the mapping $f(x) = \lambda x + (1-\lambda)y$ is continuous in $x$, for each neighborhood $U$ of $\lambda x_0 + (1-\lambda)y$ in $\mathbb{R}^n$, there exists a neighborhood $V$ of $x_0$ such that if $x \in V$, then $\lambda x + (1-\lambda)y \in U$. Since $x_0 \in \overline{A} =$ supp($\mu$), let $x \in V \cap A$. Since $A$ is starshaped relative to $y$ we have $\lambda x + (1-\lambda)y \in U \cap A$. This implies $\lambda x_0 + (1-\lambda)y \in$ supp($\mu$), so that $\overline{xy} \subseteq$ supp($\mu$). Hence, supp($\mu$) is starshaped relative to $y$. We complete the proof here. $\square$

Remark 6. In Proposition 13, the condition $\mu(y) \neq 0$ is necessary. In Example 4, we can get that supp($\mu$) = $[-2, 2]$. Let $y_0 = 3$. By Proposition 12 we have $\mu$ is f.q.s. relative to 3. Obviously $[-2, 2]$ is not starshaped relative to 3.
**Proposition 14.** \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is f.s. relative to \( y \). Then \( \text{supp}(\mu) \) is starshaped relative to \( y \).

**Proof.** According to (iii) of Theorem 3.1 and Proposition 2, this proposition is a corollary of Proposition 13. \( \square \)

**Proposition 15.** Suppose that \( \mu_1, \mu_2 \in \mathcal{F}(\mathbb{R}^n) \) are f.q-s. (respectively, f.p-s.) relative to the point \( y \in \mathbb{R}^n \). Then \( \mu_1 \cap \mu_2 \) is f.q-s. (respectively, f.p-s.) relative to the same point.

**Proof.** Suppose that \( \mu_1, \mu_2 \in \mathcal{F}(\mathbb{R}^n) \) are f.q-s. relative to \( y \in \mathbb{R}^n \). For all \( x \in \mathbb{R}^n \), we have
\[
\mu_i(\lambda x + (1 - \lambda)y) \geq \min\{\mu_i(x), \mu_i(y)\}, \quad 0 \leq \lambda \leq 1, \quad i = 1, 2,
\]
and hence
\[
(\mu_1 \cap \mu_2)(\lambda x + (1 - \lambda)y) = \min\{\mu_1(\lambda x + (1 - \lambda)y), i = 1, 2\}
\geq \min\{\mu_1(x), \mu_2(x), \mu_1(y), \mu_2(y)\}
= \min\{(\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y)\}.
\]
Suppose that \( \mu_1, \mu_2 \in \mathcal{F}(\mathbb{R}^n) \) are f.p-s. relative to the point \( y \in \mathbb{R}^n \). For all \( x \in \text{supp}(\mu_1 \cap \mu_2) \subseteq \text{supp}(\mu_1) \cap \text{supp}(\mu_2) \), we have
\[
\mu_i(\lambda x + (1 - \lambda)y) \geq \lambda \mu_i(x) + (1 - \lambda)\mu_i(y), \quad 0 \leq \lambda \leq 1, \quad i = 1, 2,
\]
and hence
\[
(\mu_1 \cap \mu_2)(\lambda x + (1 - \lambda)y) = \min\{\lambda \mu_i(x) + (1 - \lambda)\mu_i(y), i = 1, 2\}
\geq \lambda \min\{\mu_1(x), \mu_2(x)\} + (1 - \lambda)\min\{\mu_1(y), \mu_2(y)\}
= \lambda(\mu_1 \cap \mu_2)(x) + (1 - \lambda)(\mu_1 \cap \mu_2)(y).
\]
We complete the whole proof here. \( \square \)

**Proposition 16.** Suppose that \( \mu_1, \mu_2 \in \mathcal{F}(\mathbb{R}^n) \) are f.q-s. relative to the point \( y \in \mathbb{R}^n \) and \( \mu_1(y) = \mu_2(y) \). Then \( \mu_1 \cup \mu_2 \) is f.q-s. relative to the same point.

**Proof.** Suppose that \( \mu_1, \mu_2 \in \mathcal{F}(\mathbb{R}^n) \) are f.q-s. relative to \( y \in \mathbb{R}^n \). For all \( x \in \mathbb{R}^n \), we have
\[
\mu_i(\lambda x + (1 - \lambda)y) \geq \min\{\mu_i(x), \mu_i(y)\}, \quad 0 \leq \lambda \leq 1, \quad i = 1, 2.
\]
Since \( \mu_1(y) = \mu_2(y) \), we have
\[
(\mu_1 \cup \mu_2)(\lambda x + (1 - \lambda)y) = \max\{\mu_i(\lambda x + (1 - \lambda)y), i = 1, 2\}
\geq \max\{\mu_1(x), \mu_2(x)\} \cap \mu_1(y)
= \min\{(\mu_1 \cup \mu_2)(x), (\mu_1 \cup \mu_2)(y)\}.
\]
We complete the proof here. \( \square \)

**Remark 7.** The above proposition is untrue for fuzzy convex sets and is an important extension.

**Proposition 17.** If \( \mu \in \mathcal{F}(\mathbb{R}^n) \) is f.s. relative to \( y \) and is upper semi-continuous. Then \( \ker(\mu) \) is closed in \( \mathbb{R}^n \).

**Proof.** Let \( y_0 \in \ker(\mu) \). Choose \( \lambda x + (1 - \lambda)y_0 \) with \( \lambda \in [0, 1] \). Since the mapping \( f(y) = \lambda x + (1 - \lambda)y \) is continuous in \( y_0 \), for each neighborhood \( U \) of \( \lambda x + (1 - \lambda)y_0 \) in \( \mathbb{R}^n \), there exists a neighborhood \( V \) of \( y_0 \) such that if \( y \in V \), then \( \lambda x + (1 - \lambda)y \in U \).

Let \( y \in V \cap \ker(\mu) \). Since \( \mu \) is starshaped relative to \( y \) we have \( \mu(\lambda x + (1 - \lambda)y_0) \geq \mu(x) \). Owing to \( \mu \) is upper semi-continuous, we get that \( \mu(\lambda x + (1 - \lambda)y_0) \geq \mu(x) \). Hence, \( \mu \) is starshaped relative to \( y_0 \). Consequently, \( \ker(\mu) = \ker(\mu) \). \( \square \)
5. Shadow of starshaped fuzzy sets

Let $\mathcal{F}_c(\mathbb{R}^n)$ denote the collection of all fuzzy sets with the following properties: (1) $\mu$ is upper semi-continuous, (2) $\text{supp}(\mu)$ is compact, (3) $\mu$ is normal. In this section, we will yield several positive results on the shadows of starshaped fuzzy sets.

**Theorem 5.1.** Let $H$ be any ordinary hyperplane of $\mathbb{R}^n$ and $\mu$ an f.s. set relative to the point $y \in \mathbb{R}^n$. Then $S_H(\mu)$ is starshaped relative to $P(y) \in H$.

**Proof.** Since linear transformations and translations are invariant with respect to starshapedness (Propositions 8 and 9), without loss of generality, suppose that $y = (0, \ldots, 0)$ and $H = \{x \in \mathbb{R}^n : x = (x_1, x_2, \ldots, x_{n-1}, a)\}$, where $a$ is a constant. Then $P(y) = (0, 0, \ldots, 0, a)$.

Now, for any point $x \in H$, we have

$$S_H(\mu)(\lambda x - P(y)) = \sup_{\lambda \in \mathbb{R}} \{ \mu((\lambda x_1, \lambda x_2, \ldots, \lambda x_{n-1}, z)) : z \in \mathbb{R} \}$$

for each $\lambda \in [0, 1]$. Hence, $S_H(\mu)$ is starshaped relative to $P(y)$. \(\square\)

In fact, we have proved the following corollary.

**Corollary 5.1.** $P(\ker(\mu)) \subseteq \ker(S_H(\mu))$.

**Lemma 5.1.** Let $\mu \in \mathcal{F}_c(\mathbb{R}^n)$. Then $P([\mu]^2) = [S_H(\mu)]^2$ for $\alpha \in [0, 1]$.

**Proof.** Let $x \in P([\mu]^2)$. Then there exists a $y \in [\mu]^2$ such that $P(y) = x$, which implies that $S_H(\mu)(x) \geq \alpha$, that is, $x \in [S_H(\mu)]^2$.

Conversely, let $x \in [S_H(\mu)]^2$. Then $S_H(\mu)(x) = \sup_{y \in \mathbb{R}^n} \{ \mu(y) : y = P(y) = x \} \geq \alpha$.

For any positive integer $n$, we have that there is a $y_n$ such that $\mu(y_n) \geq (\alpha - 1/n)$ and $P(y_n) = x$. Since $\text{supp}(\mu)$ is compact, passing to subsequence if necessary, we may assume that the sequence $\{y_n\}$ converges to the point $y_0$. Since $\mu$ is upper semi-continuous we have that $\mu(y_0) \geq \alpha$ and $P(y_0) = x$. Consequently, $x \in P([\mu]^2)$. \(\square\)

**Lemma 5.2.** Let $\mu \in \mathcal{F}_c(\mathbb{R}^n)$. Then $[\text{conv}(\mu)]^2 = [\mu]^2$ for $\alpha \in [0, 1]$.

**Proof.** Since $[\text{conv}(\mu)]^2$ is a convex set containing $[\mu]^2$, we have that $[\text{conv}(\mu)]^2 \supseteq [\mu]^2$ for $\alpha \in [0, 1]$. For the converse part, firstly we prove that the family $\{[\text{conv}(\mu)]^2 : \alpha \in [0, 1]\}$ of subsets of $\mathbb{R}^n$ can determine a quasi-convex fuzzy set $v$, such that $[v]^2 = \text{conv}(\mu)^2$ for $\alpha \in [0, 1]$. Obviously, $\text{conv}(\mu)^2 \supseteq \mu^2$ for $0 \leq \beta \leq \alpha < 1$. Now take $\alpha_0 > 0$, $\alpha_1 \leq \alpha_2 \leq \cdots$, and $\lim_{k \to \infty} \alpha_k = \alpha_0$. We have to show that

$$\text{conv}(\mu)^2 = \bigcap_{k=1}^{\infty} \text{conv}(\mu)^{2k}.$$ 

It is clear that

$$\text{conv}(\mu)^{2k} \subseteq \bigcap_{k=1}^{\infty} \text{conv}(\mu)^{2k}.$$ 

Let $x \in \bigcap_{k=1}^{\infty} \text{conv}(\mu)^{2k}$. For each $k$, by Caratheodory theorem [26] we have that there are points $x_1^k, x_2^k, \ldots, x_{n+1}^k \in [\mu]^2$ such that $\sum_{i=1}^{n+1} \lambda_i x_i^k = x$, $\lambda_i \geq 0$ ($i = 1, \ldots, n + 1$), $\sum_{i=1}^{n+1} \lambda_i = 1$. Since $[\mu]^2$ are compact for all $k$, $[\mu]^2_{1} \supseteq \text{conv}(\mu)^{2k}$ for
\[ [\mu]^{2^n}, \ldots \text{ and the sequence } \{x_i^{k_i}\}_{k_i=1}^{\infty} \text{ is included in } [0, 1], \text{ we can choose a subsequence } \{k_j\}_{j=1}^{\infty} \text{ such that } \lim_{j \to \infty} x_i^{k_j} = x_{10} \in \bigcap_{k_{i}=1}^{\infty} [0, 1]^{2^n} \text{ and } \lim_{j \to \infty} x_{10}^{k_j} = \lambda_{i0}. \text{ By induction, we can get a subsequence } \{k_m\}_{m=1}^{\infty} \text{ such that } \lim_{m \to \infty} x_i^{k_m} = x_{10} \in \bigcap_{k_{i}=1}^{\infty} [0, 1]^{2^n} \text{ and } \lim_{m \to \infty} x_{i0}^{k_m} = \lambda_{i0}. \text{ Hence, we have that } x = \sum_{i=1}^{n+1} \lambda_{i0} x_i, \quad \lambda_{i0} \geq 0 \text{ (i = 1, ..., n + 1), } \sum_{i=1}^{n+1} \lambda_{i0} = 1, \text{ that is, } x \in \text{conv}[\mu]^{2^n}. \]

Now by Negoita–Ralescu representation theorem \[28\] we have that \( v = \text{conv}[\mu]^{2^n} \text{ for } \mu \in [0, 1]. \) Obviously, \( v \) is quasi-convex and includes \( \mu. \) By the definition of convex hull, we have \( \text{conv}(\mu) \subseteq \text{conv}[\mu]^{2^n} \) for \( \mu \in [0, 1]. \) We complete the whole proof here. \( \square \)

**Lemma 5.3** (Abe et al. [29]). Let \( S \) be a set in \( \mathbb{R}^n. \) Let \( S_i \) be defined recursively as follows:

\[
S_1 = \bigcup_{x \in S, y \in S} xy, \quad S_i = \bigcup_{x \in S_{i-1}, y \in S_{i-1}} xy.
\]

Then \( S_i = \text{conv } S \text{ if } i \text{ satisfies the inequality } 2^{i-1} \leq n + 1 \leq 2^i. \)

**Lemma 5.4** (Rockafellar [26]). Let \( S \) be a set in \( \mathbb{R}^n. \) Choose an arbitrary point \( y \in S. \) Then \( x \in \text{conv } S \iff x \text{ is contained in a finite-dimensional simplex } A \text{ having its vertices in } S \text{ and having } y \text{ as one of its vertices.} \)

**Theorem 5.2.** Let \( A \) be a starshaped set relative to \( x_0 \in \mathbb{R}^n. \) Let \( A_i \) be defined recursively as follows:

\[
A_1 = \bigcup_{x \in A, y \in A} xy, \quad A_i = \bigcup_{x \in A_{i-1}, y \in A_{i-1}} xy.
\]

Then \( A_i = \text{conv } A \text{ if } i \text{ satisfies the inequality } 2^{i-1} \leq n \leq 2^i. \)

**Proof.** Choose an arbitrary point \( x \in \text{conv } A. \) By Lemma 5.4, there are points \( x_1, \ldots, x_{r-1}, x_0 \) \( (r \leq n + 1) \) such that

\[
x \in \text{conv} \left( \{y_0\} \cup \bigcup_{i=1}^{r-1} x_i \right) = A_{r-1},
\]

where \( A_{r-1} \) is an \( r - 1 \)-dimensional simplex with its vertices \( x_1, \ldots, x_{r-1}, x_0 \in A. \) In addition, we have

\[
A_{r-1} = \text{conv} \left( \{y_0\} \cup \bigcup_{i=1}^{r-1} x_i \right)
\]

\[
= \bigcup_{\lambda \in [0, 1]} \lambda \{y_0\} + (1 - \lambda) \text{conv} \left( \bigcup_{i=1}^{r-1} x_i \right)
\]

\[
= \bigcup_{\lambda \in A_{r-2}} \lambda y_0,
\]

where \( A_{r-2} \) is an \( r - 2 \)-dimensional simplex with its vertices \( x_1, \ldots, x_{r-1} \in A. \) Since \( A \) is starshaped relative to \( y_0, \) for constructing the simplex \( A_{r-1} \) in \( \mathbb{R}^n, \) we only need to get the simplex \( A_{r-2} \) in \( \mathbb{R}^{r-1}. \) Let \( S = \{x_1, \ldots, x_{r-1}\}. \) By Lemma 5.3, we have that \( S_i = \text{conv } S = A_{r-2} \) if \( i \text{ satisfies the inequality } 2^{i-1} \leq n - 1 \leq 2^i. \) Since \( S_i \subseteq A_i, \) we have that \( A_i = \text{conv } A \text{ if } i \text{ satisfies the inequality } 2^{i-1} \leq n \leq 2^i. \) \( \square \)

**Theorem 5.3.** Let \( \mu \in \mathcal{F}_c(\mathbb{R}^n) \) with \( n \leq 2 \) and \( \mu \) be f.s. relative to \( y_0. \) Then \( S_H(\mu) = S_H(\text{conv}(\mu)) \) for any hyperplane \( H \) of \( \mathbb{R}^n. \)

**Proof.** When \( n = 1, \) by Proposition 11, we have \( \mu = \text{conv}(\mu) \) which implies \( S_H(\mu) = S_H(\text{conv}(\mu)) \) for any hyperplane \( H. \)

Suppose \( n = 2. \) For an arbitrary hyperplane \( H, \) obviously, \( S_H(\mu) \subseteq S_H(\text{conv}(\mu)). \) In other words, \( [S_H(\mu)]^2 \subseteq [S_H(\text{conv}(\mu))]^2 \) for each \( \mu \in [0, 1]. \)
Conversely, by Lemmas 5.1 and 5.2, we have

\[ [SH(\text{conv}(\mu))]^2 = P([\text{conv}(\mu)]^2) = P(\text{conv}[\mu]^2). \]

Since \([\mu]^2\) is starshaped relative to \(y_0 \in \mathbb{R}^2\), by Theorem 5.2 we have

\[ \text{conv}[\mu]^2 = [\mu]^2_1 = \bigcup_{x \in [\mu]^2, y \in [\mu)^2} xy. \]

Let \(x \in [SH(\text{conv}(\mu))]^2\). Then there exists a point \(y_1 \in \text{conv}[\mu]^2\) such that \(P(y_1) = x\). Consequently, there are two points \(w, z \in [\mu]^2\) such that \(y_1 = \lambda w + (1 - \lambda)z\). In addition, the set \(\{y \in \mathbb{R}^2 : P(y) = x\}\) actually is the line \(l\) passing through \(y_1\) and \(x\). Consequently, we know that the line \(l\) hits the sides \(\overline{wz}\) of the triangle \(\Delta_{wz y_0}\) in \(\mathbb{R}^2\). Then we claim that the line \(l\) must hit at least two sides of \(\Delta_{wz y_0}\) in \(\mathbb{R}^2\). Since the sides \(\overline{w y_0}\) and \(\overline{z y_0}\) are contained in \([\mu]^2\), we have that there exists a point \(y_2 \in [\mu]^2\) such that \(P(y_2) = x\), which implies that \(x \in P([\mu]^2) = [SH(\mu)]^2\). \(\square\)

**Remark 8.** This statement does not hold for the space \(\mathbb{R}^n\) with \(n > 2\) as shown in the following example.

**Example 7.** For simplicity, we are concerned only with the case that \(n = 3\). (When \(n > 3\), the counterexample may be similarly constructed.) Define the set \(\mu \in \mathcal{F}_c(\mathbb{R}^3)\) as

\[
\mu(x) = \begin{cases} 
1 - |x_1|, & x = (x_1, x_2, x_3) = (x_1, 0, 0) \text{ where } |x_1| \leq 1, \\
1 - |x_2|, & x = (0, x_2, 0) \text{ where } |x_2| \leq 1, \\
0 & \text{otherwise}
\end{cases}
\]

and define the hyperplane \(H\) as

\[
H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 1\}.
\]

By a simple calculation, we have that when \(x \in (0, 1)\),

\[
S_H([\text{conv(\mu)]^2} = \{(x_1, x_2, 1) \in \mathbb{R}^3 : |x_2| \leq x_1 + 1 - x \text{ and } 0 \leq x_1 \leq 1\}
\]

\[
\cup \{(x_1, x_2, 1) \in \mathbb{R}^3 : |x_2| \leq 1 - x - x_1 \text{ and } 0 \leq x_1 \leq 1 - x\}
\]

and \(S_H([\mu]^0) = H; \) when \(x \in [0, 1]\),

\[
S_H([\mu]^2) = \{(x_1, 0, 1) \in \mathbb{R}^3 : |x_1| \leq 1 - x\} \cup \{(0, x_2, 1) \in \mathbb{R}^3 : |x_2| \leq 1 - x\}
\]

and \(S_H([\mu]^0) = H. \) Obviously, \(S_H(\text{conv(\mu)} \neq S_H(\mu). \)

By Theorem 5.3, one can easily prove the following theorem.

**Theorem 5.4.** Let \(\mu_1, \mu_2 \in \mathcal{F}_c(\mathbb{R}^n)\) with \(n \leq 2\), \(\mu_1, \mu_2\) be f.s. and \(\text{conv(\mu)} = \text{conv(\mu_2)}. \) Then \(S_H(\mu_1) = S_H(\mu_2)\) for any hyperplane \(H\) of \(\mathbb{R}^n\).

**Theorem 5.5.** Let \(\mu_1, \mu_2 \in \mathcal{F}_c(\mathbb{R}^n)\) with \(n \geq 2\). If \(\mu_1\) is f.s. relative to \(\gamma_1\), \(\mu_2\) is f.s. relative to \(\gamma_2\), and \(S_H(\mu_1) = S_H(\mu_2)\) for any hyperplane \(H\) of \(\mathbb{R}^n\), then \(\mu_1 \cap \mu_2 \neq \emptyset\).

**Proof.** If \(\gamma_1 = \gamma_2\), then we have nothing to prove. Assume \(\gamma_1 \neq \gamma_2\) and \(\mu_1 \cap \mu_2 = \emptyset\).

Without loss of generality, suppose that \(\gamma_1 = (-a, 0, \ldots, 0)\) and \(\gamma_2 = (0, \ldots, 0)\), where \(a\) is a positive real number. Let \(H_1 = \{x \in \mathbb{R}^n : x = (x_1, x_2, \ldots, x_n - 1, 0)\}\), \(H_2 = \{x \in \mathbb{R}^n : x = (0, x_2, \ldots, x_n)\}\), and the corresponding orthogonal projections are \(P_1\) and \(P_2\). Then we have \(S_{H_1}(\mu_1)(\gamma_1) = 1\) and \(S_{H_2}(\mu_2)(\gamma_2) = 1\). Since for any hyperplane \(H\), \(S_{H_1}(\mu_1) = S_H(\mu_2)\), we have that there exists an point \(z \in [\mu_1]^1\) (respectively, \(w \in [\mu_2]^1\)) such that \(P_1(z) = y_2\) (respectively, \(P_2(w) = y_1\)), which implies that \(\overline{zy_2} \subseteq [\mu_1]^1\) and \(\overline{zy_1} \subseteq [\mu_2]^1\). Since \(\mu_1 \cap \mu_2 = \emptyset\) we have \([\mu_1]^1 \cap [\mu_2]^1 = \emptyset\). Hence, without loss of generality, we assume \(z = (0, 0, \ldots, 0, b)\) and \(w = (-a, 0, \ldots, 0, -c)\) where \(b\) and \(c\) are positive real numbers.
By Theorem 5.1, \( S_{H_1}(\mu_1) \) is starshaped relative to \( y_1 \) because \( y_1 = P_1(y_1) \). Furthermore, we have \( \overline{y_1y_2} \subseteq [S_{H_1}(\mu_1)]^1 \). Denote the ray originated at the point \( y_1 \) and passed though \( y_2 \) by \( \overline{y_1y_2} \). Suppose that \( v \) is the farthest point from \( y_1 \) on \( \overline{y_1y_2} \) in \( [S_{H_1}(\mu_1)]^1 \). Then there is a point \( v^* \in [\mu_1]^1 \) such that \( P_1(y_1v^*) = \overline{y_1v} \).

For the hyperplane \( H_2 \), we have \( S_{H_2}(\mu_1)(z) = 1 \). For the same reason, \( S_{H_2}(\mu_1) \) is starshaped relative to \( y_2 \) because \( y_2 = P_2(y_1) \). Suppose that \( t \) is the farthest point from \( y_2 \) on \( \overline{y_2t} \) in \( [S_{H_2}(\mu_1)]^1 \). Then there is a point \( t^* \in [\mu_1]^1 \) such that \( P_2(y_1t^*) = \overline{y_2t} \).

Since \( \overline{y_1v} \cap \overline{y_2t} = \emptyset \) and \( \overline{y_1t} \cap \overline{y_2w} = \emptyset \), we assume \( v^* = (v_1, 0, \ldots, 0, v_n) \) and \( t^* = (t_1, 0, \ldots, 0, t_n) \) where \( v_1, v_2, t_1, t_2 \) are positive real numbers. Now \( v_n \leq t_n \) and \( v_1 \geq t_1 \) because \( P_1(v^*) = (v_1, 0, \ldots, 0) = v \) and \( P_2(t^*) = (0, \ldots, 0, t_n) = t \).

Similarly, \( S_{H_2}(\mu_2) \) is starshaped relative to \( y_2 \) because \( P_2(y_2) = y_2 \). Since \( S_{H_2}(\mu_1)(t) = S_{H_2}(\mu_2)(t) \), there is a point \( q \in [\mu_2]^1 \) such that \( P_2(q) = t \), which implies that \( q = (q_1, 0, \ldots, 0, t_n) \).

We claim that \( q_1 > v_1 \). Suppose otherwise. Then we have that there are two numbers \( \lambda_1 = (t_n v_1 - v_n q_1)/(t_n v_1 + t_n a - v_n q_1), \lambda_2 = v_0 a/(t_n v_1 + t_n a - v_n q_1) \) in \([0, 1]\), so that \( \overline{\lambda_1 y_1 + (1 - \lambda_1)v^*} = \overline{\lambda_2 y_2 + (1 - \lambda_2)w} \), that is, \( \overline{y_1v^*} \cap \overline{y_2w} \neq \emptyset \). Then \( [\mu_1]^1 \cap [\mu_2]^1 \neq \emptyset \) which contradicts with \( \mu_1 \cap \mu_2 = \emptyset \). Consequently, we get \( q_1 > v_1 \). Let \( P_1(q) = (q_1, 0, \ldots, 0) = r \). We have \( S_{H_1}(\mu_1)(r) = S_{H_1}(\mu_2)(r) = 1 \). Since \( q_1 > v_1 \), we have \( \overline{y_1r} \subseteq [S_{H_1}(\mu_1)]^1 \) which contradicts with the definition of \( v \).

In conclusion, we have proved that when \( y_1 \neq y_2, \mu_1 \cap \mu_2 \neq \emptyset \) by contradiction. Thus, the theorem is proved. \( \Box \)

**Remark 9.** In [7], Liu obtained two important results on shadows of fuzzy sets.

**Theorem I (Liu [7]).** Let \( \mu_1 \) and \( \mu_2 \) be quasi-convex fuzzy sets on \( \mathbb{R}^n \) with \( n \geq 2 \) and \( S_H(\mu_1) = S_H(\mu_2) \) for each hyperplane \( H \). If \( \mu_1 \) and \( \mu_2 \) are compact relative to induced fuzzy topology, then \( \mu_1 = \mu_2 \).

**Theorem II (Liu [7]).** Let \( \mu_1 \) and \( \mu_2 \) be quasi-convex fuzzy sets on \( \mathbb{R}^n \) with \( n \geq 2 \) and \( S_H(\mu_1) = S_H(\mu_2) \) for each hyperplane \( H \). If \( \mu_1 \) and \( \mu_2 \) denote their fuzzy open-convex (closed-convex) hull, respectively, then \( \mu_1 = \mu_2 \).

Since for any \( \mu \in \mathcal{F}(\mathbb{R}^n) \), \( \mu \) is upper semi-continuous and \( \text{supp}(\mu) \) is compact, we have that \( \mu \) is compact relative to induced fuzzy topology [7]. It is easy to see that Theorem 5.5 is, in one sense, a generalization of Theorem I in [7] to f.s. sets. Liu [7] also pointed out that the inverse statement of Theorem II is not true. However, when \( n \leq 2 \), we give a positive result (Theorem 5.4) on this direction for f.s. sets.

**6. Some applications**

One application of fuzzy starshapedness can be found in Buckley and Qu [30] who consider functions mapping real numbers into the set of generalized complex fuzzy numbers, where the generalized complex fuzzy number is a fuzzy star-like set (a special kind of f.s. set) of the complex plane. And they have defined a derivative of such functions which is a generalization of the derivative of real fuzzy mappings presented by Dubois and Prade [31]. In [32], we improved the main results on the derivative of functions mapping real (or complex) numbers into the set of generalized complex fuzzy star-like numbers in [30,33].

A set \( S \) in the complex plane \( \mathbb{C} \) is connected if it cannot be represented as the union of two disjoint relatively open sets none of which is empty. A set \( S \) is said to be arcwise (or path) connected if any two points in \( S \) can be connected by an unbroken curve lying entirely within \( S \). A set \( S \) is said to be simply connected if any closed curve in \( S \) can be contracted to a point in \( S \). Equivalently, a simply connected set can be visualized as a set with no “holes”.

We will write \( z = x + iy \) for regular complex numbers and \( \mathcal{Z} \) for fuzzy sets of \( \mathbb{C} \). In [32], the generalized fuzzy complex number [30] is redefined as follows:

**Definition 6.1 (Qiu et al. [32]).** \( \mathcal{Z} \) is a generalized complex fuzzy number if and only if:

1. \( \mu(z) \mathcal{Z} \) is upper semi-continuous;
2. \( [\mathcal{Z}]^2 \) and \( \text{supp}(\mathcal{Z}) \) are compact and path connected for \( 0 < \alpha \leq 1 \); and
3. \( [\mathcal{Z}]^1 \) is nonempty.

Let \( \mathcal{Z} \) be a generalized complex fuzzy number. We will assume that \( [\mathcal{Z}]^1 = \{z_1\} \), a single point, and also \( z_1 \) belongs to the interior of \( [\mathcal{Z}]^2 \) for \( 0 < \alpha < 1 \) and the interior of \( \text{supp}(\mathcal{Z}) \). For any \( \alpha \)-level set \( [\mathcal{Z}]^2, 0 < \alpha < 1, \) draw
the ray $L(\beta)$ from $z_1$ making angle $\beta$ ($0 \leq \beta < 2\pi$) with the positive $x$-axis in the complex plane. Consider the set $S(\alpha, \beta) = L(\beta) \cap \text{boundary of } [\hat{Z}]^2$. $\hat{Z}$ is said to be star-like if and only if $S(\alpha, \beta)$ contains a single point, say $(\alpha, \beta)$, for all $0 < \alpha < 1$, and all $0 \leq \beta < 2\pi$, and $L(\beta) \cap \text{supp}(\hat{Z})$ also contains a single point, say $z(0, \beta)$, for all $0 \leq \beta < 2\pi$. We extend $\beta$ to be $0 \leq \beta \leq 2\pi$ knowing that $z(\alpha, 0) = z(\alpha, 2\pi)$ because we wish for $\beta$ to range over a closed interval.

Let $\bar{C} (\bar{C}^*)$ denote the set of generalized complex fuzzy (star-like) numbers. Now we have the following result.

**Theorem 6.1.** Let $\hat{Z} \in \bar{C}^*$. Then $\hat{Z}$ is f.s. relative to $z_1$.

**Proof.** For any $\alpha$-level set $[\hat{Z}]^2$, $0 < \alpha < 1$, let $z_2 \in [\hat{Z}]^2$. Since the ray $z_1z_2$ only meets the boundary of $[\hat{Z}]^2$ at one point, and $[\hat{Z}]^2$ is compact, we have that the line segment $z_1z_2$ is entirely contained in $[\hat{Z}]^2$. Then $[\hat{Z}]^2$ is star-shaped relative to $z_1$, for $0 < \alpha < 1$. It is obvious that $[\hat{Z}]^1 = \{z_1\}$ and $[\hat{Z}]^0 = \bar{C}$ are star-shaped relative to $z_1$. By Proposition 1, we have $\hat{Z}$ is f.s. relative to $z_1$. \square

Now let $F : (a, b) \to \bar{C}^*$ so that $F(t) = \hat{Z}(t)$ is a generalized complex fuzzy star-like number for $a < t < b$. We assume that the derivative $\dot{z}_1(t)$ of $z_1(t)$ exists for all $t$. For any $\alpha$-level set $[\hat{Z}(t)]^2$, $0 < \alpha < 1$, draw the ray $L(\beta)$ from $z_1(t)$ making angle $\beta$ ($0 \leq \beta \leq 2\pi$) with the positive $x$-axis in the complex plane. Consider the singleton $S(t, \alpha, \beta) = L(\beta) \cap \text{boundary of } [\hat{Z}(t)]^2 = \{z(t, \alpha, \beta)\}$. We may write $z(t, \alpha, \beta) = x(t, \alpha, \beta) + iy(t, \alpha, \beta)$, for all $t, \alpha, \beta$. We assume that $\dot{x}(t, \alpha, \beta)$ and $\dot{y}(t, \alpha, \beta)$ exist (these are partial derivatives with respect to $t$) for all $t, \alpha, \beta$. Let $z_1(t) = z(t, 1, \beta)$ for all $\beta$. The derivative $F'(t)$ of fuzzy complex function $F(t)$ is a fuzzy set of the complex numbers defined by its membership function $\mu(z|F'(t))$ in [30].

**Definition 6.2 (Buckley and Qu [30]).** $\mu(z|F'(t)) = \sup \{\Re z|\dot{x}(t, \alpha, \beta) + \mathcal{I}y(t, \alpha, \beta), 0 \leq \alpha \leq 1, 0 \leq \beta \leq 2\pi\}$.

In [32], we get the following result.

**Theorem 6.2 (Qiu et al. [32]).** If $\dot{x}$ and $\dot{y}$ are continuous in $\alpha$ and $\beta$, then $F'(t) \in \bar{C}$.

In [33] the authors generalized the definition of derivative in [30] to the functions mapping complex numbers into $\bar{C}^*$. Let $G \subseteq \mathbb{C}$ be a closed, bounded, simply connected domain. For function $f : G \to \bar{C}^*$, $f(z) = \hat{W}(z) \in \bar{C}^*$, we will assume that $[f(z)]^1 = [\hat{W}(z)]^1 = \{w_1(z)\}$, a single point, for $z \in G$ and also $w_1(z)$ belongs to the interior of $[\hat{W}(z)]^2$ for $0 < \alpha < 1$. We assume that $w_1(z)$ is analytic over $G$. For any $\alpha$-level set $[\hat{W}(z)]^2$, $0 < \alpha < 1$, draw the ray $L(\beta)$ from $w_1(z)$ making angle $\beta$ ($0 \leq \beta < 2\pi$) in the positive $x$-axis in the complex plane. The set

$$L(\beta) \cap \text{boundary of } [\hat{W}(z)]^2 = \{w(z, \alpha, \beta) = u(x, y, \alpha, \beta) + \mathcal{I}v(x, y, \alpha, \beta)\}$$

is a singleton, for $0 < \alpha < 1$ and $0 \leq \beta < 2\pi$, where $z = x + iy$. We now extend $\beta$ to be $0 \leq \beta \leq 2\pi$ knowing that $w(z, \alpha, 0) = w(z, \alpha, 2\pi)$ because we wish for $\beta$ to range over a closed interval. We assume that $w(z, \alpha, \beta)$ is analytic over $G$ for all $0 \leq \alpha \leq 1$ and for all $0 \leq \beta \leq 2\pi$.

**Definition 6.3 (Wu and Qiu [33]).** The derivative $f'(z)$ of $f(z)$ is a fuzzy subset of the complex numbers defined by its membership function:

$$\mu(w|f'(z)) = \sup \{\Re w|u_\alpha(x, y, \alpha, \beta) + \mathcal{I}v_\alpha(x, y, \alpha, \beta), 0 \leq \alpha \leq 1, 0 \leq \beta \leq 2\pi\},$$

for each $w \in \mathbb{C}$.

In [32], we get the following result.

**Theorem 6.3 (Qiu et al. [32]).** $f : G \to \bar{C}^*$ is derivable if and only if $f(z) = g(z) + \hat{W}_0$, where $g : G \to \mathbb{C}$ is analytic over $G$ and $\hat{W}_0 \in \bar{C}^*$ is a star-like constant such that $\hat{W}_0|1 = 0$. Thus $f'(z) = g'(z)$.

This theorem shows that fuzzy starshapedness is essential to the derivative of fuzzy complex function in fuzzy complex analysis. In [20], with the concept of f.s. set, Chaumot, Nyström and Sladoje extended the shape signature based on the distance of the boundary points from the shape centroid to the case of fuzzy sets.
Recently, in general convex optimization research, starshapedness has been attracting more and more authors’ attention [34–37]. Crespi et al. [35] studied the class of increasing-along-rays numeral functions and discussed the stability and well posedness of starshaped scalar optimization associated with increasing-along-rays property. Fang and Huang [36] generalized the notion of increasing-along-rays functions to the vectorial case and presented the relationship between increasing-along-rays property and starshaped vector optimization. It is well known that fuzzy optimization is also an active area of research in fuzzy set theory [38–42]. So we hope our work would afford some theoretical preparations to further generalize the results in [35,36] to fuzzy optimization in the future.

7. Conclusion

In this present investigation, several different types of fuzzy starshapedness have been defined as the generalizations of fuzzy convexity. They not only share some similar properties (Propositions 1, 6–10, 14, 15) with the convex case [1–4,7–9,11,19], but also extend convexity in important ways (Propositions 2–5, 11–13, 16, 17). Concerning the theorem of shadow of fuzzy convex sets, Zadeh [1,22] and Liu [7] have made some important contributions. In this paper we also get some new results on the shadow of f.s. sets. Because the concepts of fuzzy subspaces, fuzzy balanced sets and fuzzy absorbing sets are highly important in fuzzy topological vector spaces [6], the relationships between fuzzy starshapedness and those concepts will be explored in our future research.

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